The Chinese University of Hong Kong Department of Mathematics MMAT 5340 Homework 8 Please submit your assignment online on Blackboard Due at 18:00 p.m. on Monday, 31th Mar, 2025

1. Consider a Markov chain $X = (X_n)_{n \ge 0}$ with a state space $S = \{1, 2, 3, 4\}$ and the transition matrix

$$A = \begin{bmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.7 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.1 & 0.9 & 0 \end{bmatrix}$$

Find the period d(i) of each state, and which states are aperiodic?

2. Let $(X_n)_{n \in \mathbb{N}_0}$ be a simple random walk. The state space S of $(X_n)_{n \in \mathbb{N}_0}$ is the set \mathbb{Z} of all integers. We set $X_0 = 0$, and let the transition matrix P be defined by

$$P(i,j) = \begin{cases} p, & j = i+1\\ 1-p, & j = i-1\\ 0, & \text{otherwise} \end{cases}$$

for some constant $p \in (0, 1)$.

Find the period d(i) of each state, and which states are aperiodic?

3. Let $(X_n)_{n \in \mathbb{N}_0}$ be a simple random walk on \mathbb{Z}^d . The Markov chain with a transition matrix is given as follows:

$$P(x,y) = \begin{cases} \frac{1}{2d} & \text{if } \|x-y\|_1 = 1\\ 0 & \text{otherwise,} \end{cases}$$

for any $x, y \in \mathbb{Z}^d$, where $||x - y||_1 := \sum_{i=1}^d |x_i - y_i|$. First, assume that d = 2.

We divide the four directions into two groups, e.g. {up, right} and {left, down}. The Markov chain X could return to the origin 0 after 2n steps, so choose n from 2n for the location of each group as the number of up and right should be equal to the number of left and down. Finally, for each group and each $k \leq n$, we choose k from n for both groups since the number of up (right) should be equal to the number of down (left). It follows that the return probability in 2n steps is given by

$$P^{2n}(0,0) = 4^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^{2}.$$

(a) Show that

$$P^{2n}(0,0) = 4^{-2n} \binom{2n}{n}^2.$$

Hint: Consider the coefficient of $(1 + x)^n (1 + x)^n = (1 + x)^{2n}$ for each x^k , $k \in \{0, 1, \dots, 2n\}$. Then use Multinomial Theorem to deduce that $\sum_{k=0}^n {n \choose k}^2 = {2n \choose n}$.

(b) Deduce that the series $\sum_{n=1}^{\infty} P^{2n}(0,0)$ diverge, so that the random walk in dimension d=2 is recurrent.

Hint: Use Stirling's formula: $n! \approx \sqrt{2\pi n} (\frac{n}{e})^n$ for large n.

Next, we assume that d = 3.

Let us accept that, by similar arguments as in the case d = 2, the return probability of the Markov chain in 2n steps is given by

$$P^{2n}(0,0) = 6^{-2n} \binom{2n}{n} \sum_{i+j+k=n} \binom{n}{(i,j,k)^2},$$

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$.

(a) Show that

$$\sum_{i+j+k=n} \binom{n}{i,j,k} = 3^n.$$

Hint: We recall that, by Multinomial Theorem,

$$\sum_{k_1+k_2+\dots+k_m=n} \binom{n}{k_1,k_2,\dots,k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} = (x_1+x_2+\dots+x_m)^n.$$

(b) Let us consider the Gamma function $\Gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, which is defined by,

$$\Gamma(x+1) := \int_0^\infty t^x e^{-t} dt, \quad \forall x \ge 0.$$

We accept that, the second order derivative,

$$\Gamma'(x+1) = \int_0^\infty t^x e^{-t} \log t \, dt, \quad \Gamma''(x+1) = \int_0^\infty t^x e^{-t} (\log t)^2 \, dt$$

Show that the second order derivative of $x \mapsto \log(\Gamma(x+1))$ is nonnegative, and deduce that the function $x \mapsto \log(\Gamma(x+1))$ is convex.

Hint: For two functions $g(t) := \log(t)$, $h(t) \equiv 1$, we define the inner product by $\langle g, h \rangle := \int_0^\infty g(t)h(t)t^x e^{-t}dt$ and then apply the Cauchy-Schwarz inequality.

(c) Recall that

 $\Gamma(k+1) = k!$, for all positive integer k.

Deduce that if i + j + k = n, then

$$\binom{n}{i,j,k} \leq \binom{n}{n/3,n/3,n/3}.$$

Finally, use Stirling's formula to show for some constant C,

$$\binom{n}{i,j,k} \le C\frac{3^n}{n}.$$

Hint: For the first inequality, use Jensen's inequality for the convex function $\ln(n!)$.

(d) Deduce that $\sum_{n=1}^{\infty} P^{2n}(0,0) < \infty$, so that the random walk in dimension d = 3 is transient.

Hint: First write

$$P^{n}(0,0) = \binom{2n}{n} (\frac{1}{2})^{2n} \sum_{i+j+k=n} \binom{n}{(i,j,k)^{2}} (\frac{1}{3})^{2n}.$$

Then use results in (b) and (d), and find the upper bound for $\binom{2n}{n}$.